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# Symplectic alternating nil-algebras

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## Abstract

In this paper we continue developing the theory of symplectic alternating algebras that was started in [3]. We focus on nilpotency, solubility and nil-algebras. We show in particular that symplectic alternating nil-2 algebras are always nilpotent and classify all nil-algebras of dimension up to 8.

*Keywords:* Symplectic; alternating; nilpotent

AMS Subject Classification: 17D99, 08A05, 16N40

## 1 Introduction

Symplectic alternating algebras have arisen in the study of 2-Engel groups (see [1], [2]) but seem also to be of interest in their own right, with many beautiful properties. Some general theory was developed in [3].

**Definition.** Let  $F$  be a field. A *symplectic alternating algebra* over  $F$  is a triple  $L = (V, (, ), \cdot)$  where  $V$  is a symplectic vector space over  $F$  with respect to a non-degenerate alternating form  $(, )$  and  $\cdot$  is a bilinear and alternating binary operation on  $V$  such that

$$(u \cdot v, w) = (v \cdot w, u)$$

for all  $u, v, w \in V$ .

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Notice that  $(u \cdot x, v) = (x \cdot v, u) = -(v \cdot x, u) = (u, v \cdot x)$ . The multiplication by  $x$  from the right is therefore a *self-adjoint* linear operation with respect to the alternating form. We know that the dimension of a symplectic alternating algebra must be even and we will refer to a basis  $x_1, y_1, \dots, x_r, y_r$  with the property that  $(x_i, x_j) = (y_i, y_j) = 0$  and  $(x_i, y_j) = \delta_{ij}$  as a *standard basis*. We will also adopt the *left-normed* convention for multiple products. Thus  $x_1 x_2 \cdots x_n$  stands for  $(\cdots (x_1 x_2) \cdots) x_n$ . If  $x_1, x_2, \dots, x_{2r}$  is a basis for the symplectic vector space, then the alternating product is determined from the values of all triples  $(x_i x_j, x_k) = (x_j x_k, x_i) = (x_k x_i, x_j)$  for  $1 \leq i < j < k \leq 2r$ .

Given a standard basis  $x_1, y_1, \dots, x_r, y_r$  for a symplectic alternating algebra  $L$ , we can describe  $L$ , as follows. Consider the two isotropic subspaces  $Fx_1 + \cdots + Fx_r$  and  $Fy_1 + \cdots + Fy_r$ . It suffices then to write only down the products of  $x_i x_j, y_i y_j$ ,  $1 \leq i < j \leq r$ . The reason for this is that having determined these products we have determined  $(uv, w)$  for all triples  $u, v, w$  of basis vectors, since two of those are either some  $x_i, x_j$  or some  $y_i, y_j$  in which case the triple is determined from  $x_i x_j$  or  $y_i y_j$ . The only restraints on the products  $x_i x_j$  and  $y_i y_j$  come from  $(x_i x_j, x_k) = (x_j x_k, x_i) = (x_k x_i, x_j)$  and  $(y_i y_j, y_k) = (y_j y_k, y_i) = (y_k y_i, y_j)$ .

It is clear that the only symplectic alternating algebra of dimension 2 is the abelian one. Furthermore, it is easily seen that up to isomorphism there are two symplectic alternating algebras of dimension 4: one is abelian whereas the other one has the following multiplication table (see [3]).

$$\begin{array}{l}
x_1 x_2 = 0 \\
y_1 y_2 = -y_1 \\
L : \quad x_1 y_1 = x_2 \\
\quad \quad x_1 y_2 = -x_1 \\
\quad \quad x_2 y_1 = 0 \\
\quad \quad x_2 y_2 = 0.
\end{array}$$

Of course, the presentation is determined by  $x_1 x_2 = 0$  and  $y_1 y_2 = -y_1$  as the other products are consequences of these two. The symplectic alternating algebras of dimension 6 have been classified in [3], when the field has three elements: there are 31 such algebras of which 15 are simple.

As we said before, some general theory was developed in [3]. In particular it was shown that a symplectic alternating algebra is either semi simple or has an abelian ideal. In this paper we continue developing a structure theory for symplectic alternating algebras and we are motivated by the following question that was posed in [3]:

**Question.** What can one say about the structure of symplectic alternating nil-algebras? In particular, does a symplectic alternating nil-algebra have to be nilpotent?

If  $k$  is a positive integer, we say that a symplectic alternating algebra  $L$  is *nil- $k$*  if  $xy^k = 0$  for all  $x, y \in L$ . More generally, a *symplectic alternating nil-algebra* is a symplectic alternating nil- $k$  algebra for some positive integer  $k$ . Also, we define  $a \in L$  to be a *right nil- $k$  element* if  $ax^k = 0$  for all  $x \in L$  and to be a *right nil-element* if it is right nil- $k$  for some  $k$ . Similarly,  $a \in L$  is a *left nil- $k$  element* when  $xa^k = 0$  for all  $x \in L$  and a *left nil-element* if it is left nil- $k$  for some  $k$ .

Furthermore, we say that a symplectic alternating algebra is *nilpotent* if  $x_1x_2 \cdots x_n = 0$  for all  $x_1, x_2, \dots, x_n \in L$  and for some integer  $n \geq 1$ . As usual, the *nilpotency class* of  $L$  is the smallest  $c \geq 0$  such that  $x_1x_2 \cdots x_{c+1} = 0$  for all  $x_1, x_2, \dots, x_{c+1} \in L$ .

In the following, we first discuss connections between nilpotency and solubility of a symplectic alternating algebra. We will see in particular that every symplectic alternating algebra that is abelian-by-nilpotent is nilpotent. We then move to nil- $k$  elements and to symplectic alternating nil- $k$  algebras. We get a positive answer to the question above for  $k = 2$  and, when the dimension is  $\leq 8$ , also for  $k = 3$ . We finish with the classification of all nil-algebras of dimension up to 8.

## 2 Nilpotency and solubility

For subspaces  $U, V$  of a symplectic alternating algebra  $L$ , we define  $UV$  in the usual way as the subspace consisting of all linear spans of elements of the form  $uv$  where  $u \in U$  and  $v \in V$ . We define the *lower central series*  $(L^i)_{i \geq 1}$  inductively by  $L^1 = L$  and  $L^{i+1} = L^i \cdot L$ . Clearly

$$L^1 \geq L^2 \geq \dots$$

which implies in particular that every  $L^i$  is an ideal. We can also define the *upper central series*  $(Z^i(L))_{i \geq 0}$  naturally by  $Z^0(L) = \{0\}$ ,  $Z^1(L) = Z(L) = \{a \in L : ax = 0 \text{ for all } x \in L\}$  and  $Z^{i+1}(L) = \{a \in L : ax \in Z^i(L) \text{ for all } x \in L\}$ . In [3], Lemma 2.2, the author proves that the lower and the upper central series are related as follows:

$$Z^i(L) = (L^{i+1})^\perp.$$

It follows that  $Z^i(L)$  is an ideal since, in a symplectic alternating algebra,  $I^\perp$  is an ideal whenever  $I$  is an ideal (see [3], Lemma 2.1); but this also follows directly from  $Z^{i+1}(L) \cdot L \leq Z^i(L)$ . Notice also that the  $\dim(Z^i(L)) + \dim(L^{i+1}) = \dim(L)$ . We then have that  $L$  is nilpotent of class  $c \geq 0$  if and only if  $c$  is the smallest integer such that  $Z^c(L) = L$  or, equivalently,  $L^{c+1} = \{0\}$ . One more way to characterize the nilpotency in terms of the lower central series is given by the following result.

**Proposition 2.1.** *Let  $L$  be a symplectic alternating algebra. Then  $L$  is nilpotent if and only if there exists  $i \geq 1$  such that  $L^i$  is isotropic.*

*Proof.* Let  $L$  be nilpotent and denote by  $c$  its nilpotency class. Then  $L = Z^c(L) = (L^{c+1})^\perp$  and hence  $L^{c+1}$  is isotropic. Conversely, let  $L^i$  be isotropic for some  $i \geq 1$ . Then

$$(u_1 \cdots u_i, v_1 \cdots v_i) = 0$$

whenever  $u_1, \dots, u_i, v_1, \dots, v_i$  belong to  $L$ . It follows

$$(u_1, v_1 \cdots v_i u_i \cdots u_2) = 0$$

and thus  $L$  is nilpotent of class at most  $2i - 2$  since the symplectic form is non-degenerate.  $\square$

As usual, the *derived series*  $(L^{(i)})_{i \geq 0}$  is defined inductively by  $L^{(0)} = L$ ,  $L^{(1)} = L \cdot L = L^2$  and  $L^{(i+1)} = L^{(i)} \cdot L^{(i)}$ . Then

$$L^{(0)} \geq L^{(1)} \geq \dots$$

and we say that a symplectic alternating algebra  $L$  is *soluble* if there exists an integer  $n \geq 0$  such that  $L^{(n)} = \{0\}$ . The smallest  $n$  enjoying this property is then referred to as the *derived length* of  $L$ . Thus  $L$  has derived length 0 if and only if it has order one. Also, the symplectic alternating algebras with derived length at most 1 are just the abelian ones. A symplectic alternating algebra which is soluble of derived length at most 2 is said to be *metabelian*.

**Lemma 2.2.** *If  $L$  is a symplectic alternating algebra then  $L^{(i)} \subseteq L^{i+1}$ . In particular, if  $L$  is nilpotent of class  $i$  then  $L$  is soluble of derived length at most  $i$ .*

*Proof.* We argue by induction on  $i$ . The claim is obviously true when  $i = 0$  being  $L^{(0)} = L = L^1$ . Assuming  $i > 0$  and  $L^{(i)} \subseteq L^{i+1}$ , we get  $L^{(i+1)} = L^{(i)} \cdot L^{(i)} \subseteq L^{i+1} \cdot L = L^{i+2}$ , as required.  $\square$

Next result is rather odd and shows that all metabelian symplectic alternating algebras are nilpotent. It also shows that the inclusion in last lemma is not optimal.

**Proposition 2.3.** *Let  $L$  be a symplectic alternating algebra. Then  $L$  is metabelian if and only if it is nilpotent of class at most 3.*

*Proof.* We have that  $L$  is metabelian if and only if  $xy(zw) = 0$  for all  $x, y, z, w \in L$ , that is  $(xy(zw), t) = 0$  for all  $t \in L$ . This means  $0 = (xy, zwt) = (x, zwt y)$  and  $L$  is nilpotent of class at most 3.  $\square$

Not all soluble symplectic alternating algebras are however nilpotent as the following example shows.

**Example 2.4.** Consider

$$L : \begin{aligned} x_1x_2 &= 0 \\ y_1y_2 &= -y_1, \end{aligned}$$

the only nonabelian symplectic alternating algebra of dimension 4 over a field  $F$ . We have

$$Z(L) = Fx_2 \quad \text{and} \quad L^2 = Z(L)^\perp = Fx_1 + Fx_2 + Fy_1.$$

Here  $L^{(3)} = L^{(2)} \cdot L^{(2)} = Fx_2 \cdot Fx_2 = \{0\}$  and  $L$  is soluble of derived length 3 but it is not nilpotent. In fact  $y_1y_2^n = (-1)^ny_1$  for any integer  $n \geq 1$ .

However, we have the following strong generalisation of Proposition 2.3.

**Proposition 2.5.** *Let  $L$  be a symplectic alternating algebra. If  $L$  is abelian-by-(nilpotent of class  $\leq c$ ) then it is nilpotent of class at most  $2c + 1$ .*

*Proof.* Let  $I$  be an abelian ideal of  $L$  such that  $L/I$  is nilpotent of class at most  $c$ . Then  $L^{c+1} \subseteq I$  and

$$(x_1 \cdots x_{c+1} \cdot (y_1 \cdots y_{c+1}), z) = 0$$

for all  $x_1, \dots, x_{c+1}, y_1, \dots, y_{c+1}, z \in L$ . Thus

$$(x_1, y_1 \cdots y_{c+1}zx_{c+1} \cdots x_2) = 0$$

and  $L$  is nilpotent of class at most  $2c + 1$ . □

This result fails if we assume that our algebra is nilpotent-by-abelian. The example above still provides a counterexample, for  $L^2$  is nilpotent and  $L/L^2$  is abelian.

### 3 Nil-elements

Let  $L$  be a symplectic alternating algebra and  $x$  be a left nil-element of  $L$ . We say that an element  $a \in L$  has *nil- $x$  degree*  $m$  if  $m$  is the smallest positive integer such that  $ax^m = 0$ . Pick  $a \in L$  of maximal nil- $x$  degree  $k$  and let

$$V(a) = \langle a, ax, ax^2, \dots, ax^{k-1} \rangle.$$

We know that this is an isotropic subspace in  $L$  (see [3], Lemma 2.10). Then there exists  $b \in L$  such that

$$(a, b) = (ax, b) = \dots = (ax^{k-2}, b) = 0 \text{ and } (ax^{k-1}, b) = 1.$$

Since  $(a, bx^{k-1}) = (ax^{k-1}, b) = 1$ , we have that the nil- $x$  degree of  $b$  is  $k$ . Notice also that

$$(ax^r, bx^s) = (ax^{r+s}, b)$$

which is 1 if  $r + s = k - 1$  but 0 otherwise. So that the subspace

$$V(a) + V(b) = V(a) \oplus V(b) = \langle a, bx^{k-1} \rangle \oplus \langle ax, bx^{k-2} \rangle \oplus \dots \oplus \langle ax^{k-1}, b \rangle$$

is a perpendicular direct sum of hyperbolic subspaces.

Let  $W = W(a, b) = V(a) + V(b)$ . The multiplication by  $x$  from the right gives us a linear map on  $L$ . Then  $W$  is invariant under the right multiplication by  $x$  and the same is then true for the orthogonal complement  $W^\perp$ : in fact, for all  $y \in W^\perp$  and  $z \in W$  we have  $(yx, z) = -(y, zx) = 0$  as  $zx \in W$ . Now, we can take  $c \in W^\perp$  of maximal nil- $x$  degree, say  $m$ . Then, as before, we get  $d \in L$  of nil- $x$  degree  $m$  and  $W(c, d) = V(c) + V(d)$  is a perpendicular direct sum. Thus we inductively see that  $L$  splits up into a perpendicular direct sum

$$L = W(a_1, b_1) \oplus \dots \oplus W(a_n, b_n). \quad (1)$$

We will refer to such a decomposition as a *primary decomposition* of  $L$  with respect to multiplication by  $x$  from the right. We will also use the notation

$$\begin{pmatrix} a & bx^{k-1} \\ ax & bx^{k-2} \\ \vdots & \vdots \\ ax^{k-1} & b \end{pmatrix}.$$

for the subspace  $W(a, b)$ .

**Proposition 3.1.** *Let  $L$  be a symplectic alternating algebra. If  $x \in L$  is a left nil-element, then  $C_L(x)$  is even dimensional.*

*Proof.* Consider a decomposition as above with respect to right multiplication by  $x$ . We have seen that the cyclic subspaces come in pairs, say that

$$L = V(a_1) \oplus V(b_1) \oplus \dots \oplus V(a_n) \oplus V(b_n).$$

The kernel of each of these is one dimensional, hence  $C_L(x)$  has dimension  $2n$ .  $\square$

For the remainder of this section we focus on right nil-2 elements. In general, a left nil-2 element needs not to be a right nil-2 element. In Example 2.4,  $y_1$  is a left nil-2 element that is not a right nil-element. However, the converse is always true.

**Lemma 3.2.** *Let  $L$  be a symplectic alternating algebra. If  $a$  is a right nil-2 element of  $L$ , then:*

- (i)  $ayz = -azy$  for all  $y, z \in L$ ;
- (ii)  $a$  is left nil-2;
- (iii)  $C_L(a)$  is an ideal;
- (iv)  $La$  and  $Fa + La$  are abelian ideals and the latter is the smallest ideal containing  $a$ .

*Proof.* (i) We have

$$0 = a(y + z)(y + z) = (ay + az)(y + z) = ayz + azy$$

and  $ayz = -azy$ .

(ii) For all  $x \in L$ , we have  $0 = -a(a + x)^2 = xa(a + x) = xa^2$ .

(iii) Let  $x, y \in L$  and  $b \in C_L(a)$ . Then  $0 = a(x + b)^2 = ax(x + b) = axb$  which implies  $0 = (axb, y) = (a(by), x)$ . Thus  $a(by) = 0$  and  $by \in C_L(a)$ .

(iv) That  $La$  is an ideal follows immediately from  $uax = -uxa$  and of course it follows then that  $Fa + La$  is an ideal, the smallest ideal containing  $a$ . As  $a$  is left nil-2 and since  $ax(ya) = -a(ya)x = 0$ , it is clear that both the ideals are abelian.  $\square$

**Theorem 3.3.** *Let  $X$  be a set of right nil-2 elements in a symplectic alternating algebra  $L$  and denote by  $I(X)$  the smallest ideal of  $L$  containing  $X$ . Then*

$$I(X) = \sum_{a \in X} Fa + La.$$

*Furthermore, if  $|X| = c$  then  $I(X)$  is nilpotent of class at most  $c$ .*

*Proof.* Let  $a \in X$ . By Lemma 3.2 (iv) we know that  $I(a) = Fa + La$  is the smallest ideal containing  $a$  and that  $I(a)$  is abelian. It follows that  $I(X) = \sum_{a \in X} I(a)$ . Since each of these ideals is abelian it is clear that  $I(X)^{c+1} = \{0\}$ , here  $c = |X|$ .  $\square$



It follows in particular that the ideal generated by all the right nil-2 elements is always a nilpotent ideal.

## 4 Nil-2 algebras

The results concerning right nil-2 elements lead to the following characterization of symplectic alternating nil-2 algebras.

**Theorem 4.1.** *Let  $L$  be a symplectic alternating algebra. Then the following are equivalent:*

- (i)  $L$  is nil-2;
- (ii)  $C_L(x)$  is an ideal for any  $x \in L$ ;
- (iii)  $I(x)$  is abelian for any  $x \in L$ ;
- (iv) the identity  $xyz = -xzy$  holds in  $L$ ;
- (v) the identity  $x(yz) = xzy$  holds in  $L$ .

*Proof.* First we show that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). From Lemma 3.2, we know that (i) implies (ii) and (iii). To see that (iii) implies (i), take any  $a, x \in L$ . As  $I(x)$  is abelian and  $ax, x \in I(x)$ , it follows that  $ax^2 = 0$ . Finally to show that (ii) implies (i), notice that  $x \in C_L(x)$  and as  $C_L(x)$  is an ideal we also have  $ax \in C_L(x)$ . The latter gives  $ax^2 = 0$ .

We finish the proof by showing that (i) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i). The fact that (i) implies (iv) follows from Lemma 3.2. If (iv) holds, then  $x(yz) = -yzx = yxz = -xyz = xzy$  that gives us (v). Finally (i) follows from (v) by taking  $y = z$ .  $\square$

It follows from Theorem 3.3 that all symplectic alternating nil-2 algebras are nilpotent. We next analyse this in more details.

**Theorem 4.2.** *Let  $L$  be a symplectic alternating algebra over a field  $F$  of characteristic  $\neq 2$ . If  $L$  is nil-2, then  $L$  is nilpotent of class at most 3.*

*Proof.* Let  $x, y, z, t \in L$ . By Theorem 4.1,  $xy(tz) = xyzt$  and  $xy(tz) = -x(tz)y = -xzyt = xzyt = -xyzt$ . It follows that  $2xyzt = 0$  and, since  $\text{char } F \neq 2$ , we conclude that  $xyzt = 0$ .  $\square$

Moreover, the bound provided is optimal as there exists a nil-2 algebra which is nilpotent of class 3.

**Example 4.3.** Let  $F$  be any field and  $L$  be the linear span of

$$\begin{array}{ll}
x_1 = a & y_1 = tcb \\
x_2 = b & y_2 = tac \\
x_3 = c & y_3 = tba \\
x_4 = ab & y_4 = tc \\
x_5 = ca & y_5 = tb \\
x_6 = bc & y_6 = ta \\
x_7 = abc & y_7 = t.
\end{array}$$

As a symplectic vector space we let  $L = (Fx_1 + Fy_1) \oplus \cdots \oplus (Fx_7 + Fy_7)$ , a perpendicular direct sum of hyperbolic subspaces (where  $(x_i, y_i) = 1$  for  $i = 1, \dots, 7$ ). We turn this into a symplectic alternating nil-2 algebra by adding an alternating product satisfying condition (iv) of Theorem 4.1. As the identity (iv) is multilinear it suffices that  $xyz = -xzy$  whenever  $x, y, z$  are generators. The condition implies that the only non-trivial triples  $(uv, w) = (vw, u) = (wu, v)$  are

$$\begin{array}{ll}
(x_1x_2, y_4) & = 1 \\
(x_3x_1, y_5) & = 1 \\
(x_2x_3, y_6) & = 1 \\
(x_4x_3, y_7) & = 1 \\
(x_5x_2, y_7) & = 1 \\
(x_6x_1, y_7) & = 1.
\end{array}$$

Conversely one can easily check that this alternating product turns  $L$  into a symplectic alternating nil-2 algebra that is nilpotent of class 3.

**Theorem 4.4.** *Let  $F$  be a field of characteristic 2 and let  $L$  be a symplectic alternating algebra of dimension  $n = 2m$ . If  $L$  is nil-2, then  $L$  is nilpotent of class at most  $\lfloor \log_2(m+1) \rfloor$ .*

*Proof.* Let  $\{x_1, \dots, x_n\}$  be a basis of  $L$ . If  $\text{char } F = 2$ , then  $L$  is commutative and, by Theorem 4.1, it is also associative. It follows that

$$u_1 \cdots u_n = 0 \text{ for all } u_1, \dots, u_n \in L \text{ if and only if } x_1 \cdots x_n = 0.$$

But  $(x_1 \cdots x_n, x_i) = 0$  for any  $i \in \{1, \dots, n\}$ . Hence  $x_1 \cdots x_n = 0$  and  $L$  is nilpotent of class at most  $n - 1$ . So, if we denote by  $c$  the nilpotency class of  $L$ , then  $c < n$ . Since the class is  $c$  there is a non-zero product  $x_{i_1} \cdots x_{i_c}$  and without loss of generality we can suppose that  $x_1 \cdots x_c \neq 0$ . Now, let

$$x_I = x_{i_1} \cdots x_{i_r}$$

for any  $I = \{i_1, \dots, i_r\} \subseteq \{1, \dots, c\}$  and let

$$X = \{x_I : \emptyset \subset I \subseteq \{1, \dots, c\}\}.$$

We prove that  $X$  is a linearly independent subset of  $L$ . Assume

$$\alpha_1 x_{I_1} + \dots + \alpha_m x_{I_m} = 0$$

where  $m \leq 2^c - 1$  and  $|I_1| \leq \dots \leq |I_m|$ . Let  $\alpha_j$  be the least non zero coefficient and  $J = \{1, \dots, c\} \setminus I_j$ . Then, multiplying by  $\prod_{k \in J} x_k$ , we get

$$\alpha_j x_1 \cdots x_c = 0$$

and thus  $x_1 \cdots x_c = 0$  which is a contradiction. Thus  $X$  is linearly independent and  $|X| = 2^c - 1$ . Hence  $2^c - 1 \leq 2m$  and  $2^c < 2m + 2$ . Then  $c < \log_2(2(m + 1)) = 1 + \log_2(m + 1)$  and so  $c \leq \log_2(m + 1)$ , as we claimed.  $\square$

Indeed, the bound we have just got is the best possible, as shown in the following construction:

**Example 4.5.** Let  $F$  be the field with 2 elements and let  $r > 3$ . There exists a symplectic alternating nil-2 algebra  $L$  over  $F$  of dimension  $2(2^{r-1} - 1)$  which is nilpotent of class  $r - 1$ . In fact, define  $L$  to be the linear span of all monomials in  $x_1, \dots, x_r$  with no repeated entries and of weight less than  $r$ . Then  $L$  has dimension  $2^r - 2$  over  $F$ . Let

$$(x_{i_1} \dots x_{i_n}, x_{j_1} \dots x_{j_m}) = 0$$

except if  $n + m = r$  and  $\{i_1, \dots, i_n, j_1, \dots, j_m\} = \{1, \dots, r\}$ , and 1 otherwise. This gives a symplectic vector space. Let

$$x_{i_1} \dots x_{i_n} \cdot x_{j_1} \dots x_{j_m} = x_{i_1} \dots x_{i_n} x_{j_1} \dots x_{j_m}$$

if  $i_1, \dots, i_n, j_1, \dots, j_m$  are distinct and  $\{i_1, \dots, i_n, j_1, \dots, j_m\} \subset \{1, \dots, r\}$ , and 0 otherwise. Then  $L$  is a symplectic alternating algebra that is nilpotent of class  $r - 1$ . Since  $L$  is commutative and associative, it is also nil-2.

## 5 Nil-3 algebras

In this section we describe some general properties of a symplectic alternating nil-3 algebra  $L$ .

**Lemma 5.1.** *For any  $x, y_i, z \in L$  the following identities hold:*

$$(i) \sum_{\sigma \in S_3} xy_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)} = 0;$$

$$(ii) \sum_{\sigma \in S_2} xy_{\sigma(1)}y_{\sigma(2)}z + xy_{\sigma(1)}(zy_{\sigma(2)}) + x(zy_{\sigma(1)}y_{\sigma(2)}) = 0.$$

*Proof.* The proof of (i) is straightforward. To see why (ii) holds notice that, for any  $u \in L$ , from (i) we have

$$\begin{aligned} 0 &= \left( \sum_{\sigma \in S_2} xy_{\sigma(1)}y_{\sigma(2)}u + xy_{\sigma(1)}uy_{\sigma(2)} + xuy_{\sigma(1)}y_{\sigma(2)}, z \right) \\ &= \sum_{\sigma \in S_2} (xy_{\sigma(1)}y_{\sigma(2)}, zu) + (xy_{\sigma(1)}, zy_{\sigma(2)}u) + (x, zy_{\sigma(2)}y_{\sigma(1)}u) \\ &= - \left( \sum_{\sigma \in S_2} xy_{\sigma(1)}y_{\sigma(2)}z + xy_{\sigma(1)}(zy_{\sigma(2)}) + x(zy_{\sigma(2)}y_{\sigma(1)}), u \right). \end{aligned}$$

□

In the following we will use the notation

$$x\{y_1, y_2, y_3\}$$

for the first sum in Lemma 5.1 and similarly

$$x\{y_1, y_2\} = xy_1y_2 + xy_2y_1.$$

**Lemma 5.2.** *For any  $x, y, z \in L$  the following hold:*

$$(i) \quad yx^2y = -yxyx \in Lx;$$

$$(ii) \quad \text{if } zx^2y = 0 \text{ then } yx^2z \in Lx;$$

$$(iii) \quad yx^2(zx^2) \in Lx \cap C_L(x);$$

$$(iv) \quad \text{if } yx^2(zx^2) = 0 \text{ then } yx^2(zx) \in Lx \cap C_L(x).$$

*Proof.* (i) First we have

$$0 = y(x+y)^3 = yx(x+y)^2 = (yx^2 + yxy)(x+y) = yx^2y + yxyx.$$

(ii) Assume  $zx^2y = 0$ . Then we get

$$\begin{aligned} 0 &= x\{x, y, z\} \\ &= xy\{x, z\} + xz\{x, y\} \\ &= yxzx + xzyx + xzyx \end{aligned}$$

that gives  $yx^2z \in Lx$ .

(iii) We see that

$$0 = -x\{x, yx, zx^2\} = yx^2\{x, zx^2\} = yx^2(zx^2)x.$$

Then also

$$\begin{aligned} 0 &= x\{x, y, zx^2\} \\ &= xy\{x, zx^2\} \\ &= xyx(zx^2) + xy(zx^2)x \end{aligned}$$

that implies  $yx^2(zx^2) \in Lx \cap C_L(x)$ .

(iv) Let  $yx^2(zx^2) = 0$ . Since

$$0 = x\{x, yx^2, z\} = xz(yx^2)x,$$

it follows

$$yx^2(zx)x = 0.$$

Notice also

$$\begin{aligned} 0 &= x\{x, y, zx\} \\ &= xy\{x, zx\} + x(zx)\{x, y\} \\ &= xyx(zx) + xy(zx)x + x(zx)yx. \end{aligned}$$

Thus  $yx^2(zx) \in Lx \cap C_L(x)$ . □

## 6 Classification of nil-algebras of dimension $\leq 8$

Before embarking on the classification of the symplectic alternating nil-algebras of dimension  $\leq 8$ , we prove the following result.

**Proposition 6.1.** *If  $L$  is a symplectic alternating nil- $k$  algebra, then  $\dim(L) \geq 2(k+1)$ .*

*Proof.* Suppose by contradiction  $\dim(L) = 2k$  and take  $x \in L$  which is not left nil- $(k-1)$ . By (1), there is only one possible primary decomposition for the multiplication by  $x$  from the right. This is

$$\begin{pmatrix} a & bx^{k-1} \\ ax & bx^{k-2} \\ \vdots & \vdots \\ ax^{k-1} & b \end{pmatrix}.$$

It is easy to see that  $x = cx^{k-1}$  for some  $c \in L$ . Then  $0 = x(-cx^{k-2})^k = x$ , which is impossible. □

As a consequence, all the nonabelian nil-algebras of dimension  $\leq 8$  are the nil-2 algebras of dimension either 6 or 8 and the nil-3 of dimension 8.

## 6.1 Nil-2 algebras of dimension 6

Let  $L$  be a symplectic alternating nil-2 algebra of dimension 6 over a field  $F$ . Assume that  $L$  is not abelian and let  $x \in L \setminus Z(L)$ . Because of (1), we have that the only primary decomposition of  $L$  with respect to multiplication by  $x$  from the right is

$$\begin{pmatrix} a & bx \\ ax & b \end{pmatrix} \oplus \begin{pmatrix} c & d \end{pmatrix}$$

where  $cx = dx = 0$ .

By Theorem 4.1,  $axc = -xac = xca = 0$  and similarly  $ax$  commutes with  $d, a, ax, bx$ . As  $C_L(ax)$  is even dimensional, it follows that  $ax$  commutes also with  $b$  and thus  $ax \in Z(L)$ . Similarly  $bx \in Z(L)$  and  $Lx \subseteq Z(L)$ . Of course this is also true if  $x \in Z(L)$ . We have thus shown that  $Ly \subseteq Z(L)$  for all  $y \in L$  and thus  $L$  is nilpotent of class 2.

Now we have

$$x = \alpha ax + \beta bx + u$$

for some  $\alpha, \beta \in F$  and  $u \in Fc + Fd$ . As  $x \notin Lx$  we must have that  $u$  is nontrivial. Also  $au = ax$  and  $bu = bx$ . We can thus, without loss of generality, replace  $x$  by  $u$  and suppose that  $x$  is orthogonal to  $a, ax, b, bx$ . Next we turn to  $ab$ . Notice that  $ab$  is orthogonal to  $a, b, ax, bx$  and  $(x, ab) = (-bx, a) = (a, bx) = 1$ . Hence we have the primary decomposition

$$\begin{pmatrix} a & bx \\ ax & b \end{pmatrix} \oplus \begin{pmatrix} x & ab \end{pmatrix}$$

with respect to multiplication by  $x$  from the right. The structure is now completely determined. So there is just one nonabelian nil-2 algebra of dimension 6.

## 6.2 Nil-2 algebras of dimension 8

Let  $L$  be a symplectic alternating nil-2 algebra of dimension 8 over a field  $F$ . Assume that  $L$  is not abelian and let  $x \in L \setminus Z(L)$ . We cannot have  $x \in Lx$  as this would imply that  $x = xz$  for some  $z \in L$  and then  $x = xz^2 = 0$ . By (1), this implies that there is only one possible primary decomposition of  $L$  with respect to multiplication by  $x$  from the right. This is

$$\begin{pmatrix} a & bx \\ ax & b \end{pmatrix} \oplus \begin{pmatrix} c & d \end{pmatrix} \oplus \begin{pmatrix} e & f \end{pmatrix}$$

where  $cx = dx = ex = fx = 0$ .

By Theorem 4.1,  $axc = -xac = xca = 0$  and similarly we see that  $ax$  commutes with  $d, e, f, bx$  as well as, of course, with  $a$  and  $ax$ . Since  $C_L(ax)$  is even dimensional, it follows that  $ax$  commutes also with  $b$  and  $ax \in Z(L)$ . The same argument shows that  $bx \in Z(L)$ . So  $Lx \subseteq Z(L)$  and obviously

this is also true if  $x \in Z(L)$ . We have thus shown that  $Ly \subseteq Z(L)$  for all  $y \in L$  and  $L$  is nilpotent of class 2. Now we have that

$$x = \alpha ax + \beta bx + u$$

for some  $\alpha, \beta \in F$  and for  $u \in Fc + Fd + Fe + Ff$ . As  $x$  cannot be in  $Lx$  we must have that  $u$  is nontrivial. Now  $au = ax$  and  $bu = bx$  so we can, without loss of generality, replace  $x$  by  $u$  and so we can suppose that  $x$  is orthogonal to  $a, b, ax, bx$ . Next consider the element  $ab$ . We have that  $ab$  is orthogonal to  $a, b$  and as  $ab \in Z(L)$ , we also have that  $ab$  is orthogonal to  $ax$  and  $bx$ . Furthermore  $(x, ab) = (-bx, a) = (a, bx) = 1$ . So we have a primary decomposition

$$\begin{pmatrix} a & bx \\ ax & b \end{pmatrix} \oplus \begin{pmatrix} x & ab \end{pmatrix} \oplus \begin{pmatrix} c & d \end{pmatrix} \quad (2)$$

with  $cx = dx = 0$ . But now  $Fa + Fax + Fbx + Fb + Fx + Fab$  is invariant under multiplication by  $a$  and  $b$ . It follows that its orthogonal complement,  $Fc + Fd$ , is also invariant under multiplication by  $a$  and  $b$ . The only possibility then is that  $ca = da = cb = db = 0$ . Notice, finally, that  $cd$  is orthogonal to  $a, ax, b, bx, x, ab$  as well as to  $c, d$  and thus  $cd = 0$ . The structure of  $L$  is thus determined. All triples  $(uv, w)$  involving  $ax, bx, ab, c, d$  are trivial and  $(ax, b) = (xb, a) = (ba, x) = 1$ . So there is only one nonabelian nil-2 algebra of dimension 8.

### 6.3 Nil-3 algebras of dimension 8

Let  $L$  be a symplectic alternating nil-3 algebra of dimension 8 over a field  $F$ . Suppose that  $x \in L$  is not left nil-2. By (1), there is only one possible primary decomposition for the multiplication by  $x$  from the right. This is

$$L = \begin{pmatrix} a & bx^2 \\ ax & bx \\ ax^2 & b \end{pmatrix} \oplus \begin{pmatrix} u & t \end{pmatrix}$$

where  $ux = tx = 0$ .

**Lemma 6.2.** *The following properties hold:*

- (i)  $Lx^2$  is abelian;
- (ii)  $Lx^2(Lx) \subseteq Lx^2$ ;
- (iii)  $ax^2(ax) = -ax^2ax$  and  $bx^2(bx) = -bx^2bx$ ;
- (iv) if  $bx^2(ax) = 0$  then  $ax^2(ax) = rbx^2$  for some  $r \in F$ ;
- (v) if  $ax^2(bx) = 0$  then  $bx^2(bx) = sax^2$  for some  $s \in F$ .

*Proof.* (i) As  $Lx \cap C_L(x) = Lx^2$ , it follows from Lemma 5.2 (iii) that  $ax^2(bx^2) \in Lx^2 = Fax^2 \oplus Fbx^2$ . Suppose

$$ax^2(bx^2) = \alpha ax^2 + \beta bx^2$$

for some  $\alpha, \beta \in F$ . Then

$$0 = ax^2(bx^2)^3 = \alpha^3 ax^2 + \alpha^2 \beta bx^2$$

implies  $\alpha = 0$  and

$$0 = bx^2(ax^2)^3 = -\beta^3 bx^2$$

gives  $\beta = 0$ . Thus  $ax^2(bx^2) = 0$  and  $Lx^2$  is abelian.

(ii) This follows by (i) and Lemma 5.2 (iv), since  $Lx \cap C_L(x) = Lx^2$ .

(iii) We have

$$0 = -x\{a, x, ax\} = ax\{x, ax\} + ax^2\{a, x\} = ax^2(ax) + ax^2ax$$

and similarly  $0 = bx^2(bx) + bx^2bx$ .

(iv) By (ii), we know that

$$ax^2(ax) = sax^2 + rbx^2$$

for some  $r, s \in F$ . Then

$$0 = -x(ax)^3 = ax^2(ax)^2 = s^2 ax^2 + srbx^2$$

implies  $s = 0$  and hence  $ax^2(ax) = rbx^2$ .

We get (v) in the same manner. □

Notice that the following result holds with the roles of  $a$  and  $b$  interchanged.

**Lemma 6.3.** *If  $ax^2(ax) = rbx^2$  for some  $r \in F$ , then  $ax^2(bx) = 0$ . Furthermore,  $ax^2 \in Z(L)$  when  $r = 0$ .*

*Proof.* By (i) of Lemma 5.2,  $ax^2a \in Lx$ . As  $(ax^2a, a) = 0$  and

$$(ax^2a, ax) = -(ax^2(ax), a) = r,$$

we have

$$ax^2a = \alpha ax + \beta ax^2 - rbx$$

for some  $\alpha, \beta \in F$ . Then

$$ax^2ax = \alpha ax^2 - rbx^2.$$

But  $ax^2ax = -ax^2(ax) = -rbx^2$  by Lemma 6.2 (iii), thus  $\alpha ax^2 = 0$ . It follows that  $\alpha = 0$  and

$$ax^2a = \beta ax^2 - rbx,$$



so that  $ax^2a$  is orthogonal to  $bx$  and thus  $ax^2(bx)$  is orthogonal to  $a$ . However,  $ax^2(bx) \in Lx^2$  by (ii) of Lemma 6.2, hence

$$ax^2(bx) = \gamma ax^2$$

for some  $\gamma \in F$ . Moreover  $0 = ax^2(bx)^3 = \gamma^3 ax^2$ , hence  $\gamma = 0$  and  $ax^2(bx) = 0$ .

Now assume  $r = 0$ . Then

$$ax^2a = \beta ax^2$$

and we have

$$0 = ax^2a^3 = \beta^3 ax^2$$

which gives  $\beta = 0$  and

$$ax^2a = 0.$$

We now turn to  $ax^2u$  and  $ax^2t$ . They both lie in  $Lx$  by (ii) of Lemma 5.2 and are orthogonal to  $a, ax, bx$ . If  $\beta = (ax^2u, b)$  and  $\gamma = (ax^2t, b)$ , we have

$$ax^2u = \beta ax^2 \quad \text{and} \quad ax^2t = \gamma ax^2.$$

Then, as before, we get  $\beta = \gamma = 0$ . We have thus seen that  $ax^2$  commutes with  $a, ax, ax^2, bx, bx^2, u, t$  and, as the dimension of  $C_L(ax^2)$  is even, it follows that  $ax^2b = 0$  and  $ax^2 \in Z(L)$ .  $\square$

**Corollary 6.4.** *Let  $y, z \in L$ . If  $yz^2(yz) = 0$  then  $yz^2 \in Z(L)$ .*

*Proof.* If  $yz^2 = 0$ , this is obvious. Otherwise this follows from Lemma 6.3 with  $y$  in the role of  $a$  and  $z$  in the role of  $x$ .  $\square$

**Remark 6.5.** *In particular if  $yz^2(yz) = 0$  for all  $y, z \in L$ , then  $Lz^2 \subseteq Z(L)$ .*

Furthermore, we have:

**Lemma 6.6.**  $Z(L) \cap Lx^2 \neq \{0\}$ .

*Proof.* If  $ax^2(ax) = 0$ , then  $ax^2 \in Z(L)$  by the previous lemma. So we may assume  $ax^2(ax) \neq 0$ . By Lemma 6.2 (ii), the multiplication by  $ax$  from the right gives us a linear operator on  $Lx^2$  that is a nil operator and so with a nontrivial kernel. This means that we have

$$(b + \alpha a)x^2(ax) = 0$$

for some  $\alpha \in F$ . Without loss of generality we can replace  $b$  by  $b + \alpha a$  and thus assume that

$$bx^2(ax) = 0.$$

By Lemma 6.2 (iv) we have  $ax^2(ax) = rbx^2$  for some  $r \in F \setminus \{0\}$  and hence  $ax^2(bx) = 0$  by Lemma 6.3. Then (v) of Lemma 6.2 gives that there exists  $s \in F$  such that  $bx^2(bx) = sax^2$ . This implies

$$0 = bx^2(ax + bx)^3 = rs^2ax^2$$

and we get  $s = 0$ . It follows  $bx^2(bx) = 0$  and  $bx^2 \in Z(L)$  again applying Lemma 6.3.  $\square$

We now turn to the structure of  $L$ . This is determined by the value of all triples  $(vz, w) = (zw, v) = (wv, z)$  where  $v, z, w$  are pairwise distinct basis vectors. As any such triples has either two vectors from  $\{a, ax, ax^2, b, bx, bx^2\}$  or two vectors from  $\{u, t\}$ , we only need to determine  $ut$  and the products of any two elements from  $\{a, ax, ax^2, b, bx, bx^2\}$ .

According with Lemma 6.6, we will assume

$$bx^2 \in Z(L). \quad (3)$$

Then we also have

$$ax^2(ax) = rbx^2 \quad \text{and} \quad ax^2(bx) = 0 \quad (4)$$

by Lemma 6.2 (iv) and Lemma 6.3, respectively.

**Step 1.** *We can assume that  $ax^2b = 0$  and  $ax^2a = -rbx$ .*

*Proof.* By Lemma 5.2, (ii) and (i),  $ax^2b$  and  $ax^2a$  are in  $Lx$ . Also  $ax^2b$  is orthogonal to  $ax, b, bx$  and

$$ax^2b = \alpha bx^2$$

for  $\alpha = -(ax^2b, a)$ . If  $r = 0$ , then Lemma 6.3 implies  $ax^2 \in Z(L)$  and so  $ax^2b = 0$ . Let  $r \neq 0$ , then  $ax^2(b - \frac{\alpha}{r}ax) = 0$ . Replacing  $b$  by  $b - \frac{\alpha}{r}ax$ , we can assume that  $ax^2b = 0$ . One can check that (3) and (4) still hold.

Next, we have that  $ax^2a$  is orthogonal to  $a, b, bx$  and

$$(ax^2a, ax) = -(ax^2(ax), a) = -r(bx^2, a) = r.$$

Thus  $ax^2a = -rbx$ .  $\square$

Suppose now that  $x = y + z$  with  $y \in \langle a, ax, ax^2, b, bx, bx^2 \rangle$  and  $z \in \langle u, t \rangle$ . Then  $0 = yx$  and thus  $y \in Lx^2$ . Notice that  $z \neq 0$  since otherwise  $x = y = cx^2$  for some  $c \in L$  and  $0 = x(-cx)^3 = x$ . Without loss of generality, we can suppose that  $z = u$ . Hence

$$x = u + \alpha ax^2 + \beta bx^2$$

for some  $\alpha, \beta \in F$ .

Let us calculate the effect of multiplying with

$$u = x - \alpha ax^2 - \beta bx^2.$$

Firstly, we have

$$ut = xt - \alpha ax^2 t.$$

However,  $ax^2 t \in Lx$  by Lemma 5.2 (ii) and is orthogonal to  $a, ax, b, bx$ . Thus  $ax^2 t = 0$  and

$$ut = xt.$$

Recall that  $bx^2 \in Z(L)$  and that  $ax^2 b = ax^2(bx) = 0$ , whereas  $ax^2 a = -rbx$  and  $ax^2(ax) = rbx^2$ . Using this, we see that

$$au = ax + \alpha ax^2 a = ax - \alpha rbx$$

and

$$\begin{aligned} au^2 &= (ax - \alpha rbx)(x - \alpha ax^2 - \beta bx^2) \\ &= ax^2 + \alpha ax^2(ax) - \alpha rbx^2 \\ &= ax^2 + \alpha rbx^2 - \alpha rbx^2 \\ &= ax^2. \end{aligned}$$

One also sees that  $bu = bx$  and  $bu^2 = bx^2$ . Replacing  $x$  by  $u$  and  $a, ax, ax^2, b, bx, bx^2$  by  $a, au, au^2, b, bu, bu^2$ , we still have a decomposition into hyperbolic subspaces. One can now check that (3), (4) and Step 1 are still valid with  $x$  replaced by  $u$ . So without loss of generality we can assume that  $u = x$ . We thus have a primary decomposition

$$L = \begin{pmatrix} a & bx^2 \\ ax & bx \\ ax^2 & b \end{pmatrix} \oplus \begin{pmatrix} x & t \end{pmatrix}$$

where

$$xt = 0. \tag{5}$$

**Step 2.**  $ax(bx) = 0$ .

*Proof.* From  $ax^2 b = 0$ , we get

$$0 = -x\{a, b, x\} = ax\{b, x\} + bx\{a, x\} = axbx + bxax. \tag{6}$$

Since the values

$$(axb, b), (axb, ax), (axb, ax^2), (axb, bx^2)$$

and

$$(bxa, a), (bxa, bx), (bxa, ax^2), (bxa, bx^2)$$

are all trivial, we have

$$axb = \alpha ax + y, \quad y \in Fbx^2 + Fx + Ft \quad (7)$$

and

$$bxa = \beta bx + z, \quad z \in Fax^2 + Fx + Ft, \quad (8)$$

respectively. By (6), (7) and (8), it follows that

$$\alpha ax^2 = axbx = -bxa x = -\beta bx^2$$

which implies  $\alpha = \beta = 0$ . Hence  $(axb, bx) = (bxa, ax) = 0$  and thus

$$(ax(bx), a) = (ax(bx), b) = 0.$$

Clearly,  $ax(bx)$  is also orthogonal to  $ax, bx, ax^2, bx^2, x$  and thus

$$ax(bx) = \alpha x$$

for some  $\alpha \in F$ . But we have

$$\begin{aligned} 0 &= -x\{a, ax, bx\} \\ &= ax\{ax, bx\} + ax^2\{a, bx\} + bx^2\{a, ax\} \\ &= ax(bx)(ax) + ax^2a(bx) \\ &= ax(bx)(ax) - r(bx)^2 \\ &= ax(bx)(ax). \end{aligned}$$

Then

$$0 = ax(bx)(ax) = \alpha x(ax) = -\alpha ax^2$$

and  $\alpha = 0$ . □

**Step 3.** We can assume that  $bx b = 0$  and  $axa = rb$ .

*Proof.* Let us first consider  $bx b$ . It is orthogonal to  $ax, ax^2, b, bx, bx^2, x$ . We then have

$$bx b = \alpha bx^2 + \beta x$$

where  $\alpha = -(bx b, a)$  and  $\beta = (bx b, t)$ . Since

$$0 = xb^3 = -\beta xb,$$

we get  $\beta = 0$ . It follows that

$$0 = bx(b - \alpha x).$$

Replacing  $b$  by  $b - \alpha x$  and  $t$  by  $t - \alpha ax^2$  respectively, (3), (4), (5) and the previous steps still hold. Thus we can assume  $bx b = 0$ .

We turn to  $axa$ . It is clear that  $axa$  is orthogonal to  $a, ax, bx, bx^2, x$  and that

$$(axa, ax^2) = (ax^2, a(ax)) = (ax^2(ax), a) = r(bx^2, a) = -r.$$

Suppose  $(axa, b) = \alpha$  and  $(axa, t) = \beta$ . Then

$$axa = \alpha ax^2 + rb + \beta x. \quad (9)$$

We next show that  $axa(bx) \in Lx$  and in order to do this we prove that  $a(bx)x = 0$ . That this is sufficient follows from

$$0 = a\{a, x, bx\} = ax\{a, bx\} + a(bx)\{a, x\} = axa(bx) + a(bx)ax + a(bx)xa.$$

As  $ax(bx) = 0$ , by (8) we know that  $a(bx) \in Fax^2 + Fx + Ft$ . But

$$(a(bx), b) = 0 \quad \text{and} \quad (a(bx), x) = -1,$$

and thus

$$a(bx) = \gamma x + t \quad \text{and} \quad a(bx)x = 0. \quad (10)$$

Let  $axa(bx) = \alpha_1 ax + \alpha_2 ax^2 + \beta_1 bx + \beta_2 bx^2$ . Since

$$(axa(bx), a) = (axa(bx), b) = (axa(bx), ax) = (axa(bx), bx) = 0,$$

$axa(bx)$  is trivial and, by (9), we get

$$0 = axa(bx) = -\beta bx^2.$$

Thus  $\beta = 0$  and  $ax(a - \alpha x) = rb$ . If we replace  $a$  by  $a - \alpha x$  and  $t$  by  $t + \alpha bx^2$ , then (3), (4), (5) and all the previous steps hold. So we can assume that  $axa = rb$ .  $\square$

**Step 4.**  $axb = t$  and  $bxa = -t$ .

*Proof.* We first consider  $axt$  which is clearly orthogonal to  $x$  and  $t$ . As the product of  $ax$  with  $a, ax, ax^2, bx, bx^2$  is orthogonal to  $t$ ,  $axt$  is also orthogonal to  $a, ax, ax^2, bx, bx^2$ . Hence, for some  $\alpha \in F$ ,

$$axt = \alpha ax^2 \quad \text{and} \quad ax(t - \alpha x) = 0.$$

Replacing  $t$  by  $t - \alpha x$  we can assume that

$$axt = 0.$$

It follows that  $(axb, t) = 0$ , thus  $axb$  is orthogonal to  $t$ . As the products of  $ax$  with  $a, ax, bx, ax^2, bx^2$  are orthogonal to  $b$ , we have that  $axb$  is orthogonal to  $t, a, ax, bx, ax^2, bx^2, b$ . Also  $(axb, x) = -1$  and so

$$axb = t.$$

We now turn to  $bx a$ . By (10), we know that

$$bx a = -t - \gamma x.$$

Since

$$\begin{aligned} 0 &= -x(a+b)^3 \\ &= (ax+bx)(a+b)^2 \\ &= (axa+axb+bx a)(a+b) \\ &= (rb+t-t-\gamma x)(a+b) \\ &= -rab+\gamma ax+\gamma bx, \end{aligned}$$

we get

$$0 = (-rab + \gamma ax + \gamma bx, bx) = \gamma.$$

Thus  $bx a = -t$ . □

**Step 5.** *We can assume that  $ab = 0$ .*

*Proof.* Clearly,  $ab$  is orthogonal to  $a, b$  and, since  $ax^2, bx, bx^2$  commute with  $b$ , we have that  $ab$  is also orthogonal to  $ax^2, bx, bx^2$ . As  $bx$  is orthogonal to  $a$  we also have  $ab$  orthogonal to  $x$ . Then

$$(ab, ax) = -(b, axa) = -(b, rb) = 0$$

and the only generator left is  $t$ . Hence

$$ab = \alpha x$$

for some  $\alpha \in F$ .

We consider two cases. Suppose first that  $yz^2(yz) = 0$  for all  $y, z \in L$ . Then  $r = 0$  and by Remark 6.5

$$\alpha xb = ab^2 \in Z(L)$$

which is absurd except if  $\alpha = 0$ . Hence  $ab = 0$  in this case.

If the identity  $yz^2(yz) = 0$  does not hold for all  $y, z \in L$ , without loss of generality we can assume  $ax^2(ax) = rbx^2$  with  $r \neq 0$ . Thus

$$0 = ba^3 = \alpha axa = \alpha rb$$

implies  $\alpha = 0$  and hence  $ab = 0$  also in this case. □

As candidates for our examples we thus have a one parameter family of symplectic alternating algebras

$$L(r) = \begin{pmatrix} a & bx^2 \\ ax & bx \\ ax^2 & b \end{pmatrix} \oplus \begin{pmatrix} x & t \end{pmatrix}.$$

Notice that  $t \in Z(L(r))$  since  $vt$  is orthogonal to  $x, t$  and  $(vt, w) = -(vw, t) = 0$  for all  $v, w \in \{a, ax, ax^2, b, bx, bx^2\}$ : the only nontrivial products not involving  $x$  are

$$\begin{aligned} axa &= rb \\ ax^2a &= -rbx \\ ax^2(ax) &= rbx^2 \\ axb &= t \\ bxa &= -t. \end{aligned}$$

It remains to check that  $L(r)$  is nil-3.

**Proposition 6.7.**  *$L(r)$  is a nil-3 algebra for all  $r \in F$ .*

*Proof.* Let  $z = \alpha_1 a + \alpha_2 ax + \alpha_3 ax^2 + \beta_1 b + \beta_2 bx + \gamma x$ . It suffices to show that  $yz^3 = 0$  for the basis elements  $a, ax, ax^2, b, bx, x$ . Using the description of  $L(r)$ , we have  $bxz^2 = (-\alpha_1 t + \gamma bx^2)z = 0$  and then:

$$\begin{aligned} az^3 &= (-\alpha_2 rb + \alpha_3 rbx + \beta_2 t + \gamma ax)z^2 \\ &= (-\alpha_2 rb + \gamma ax)z^2 \\ &= (\alpha_2^2 rt - \alpha_2 \gamma rbx + \gamma \alpha_1 rb - \gamma \alpha_3 rbx^2 + \gamma \beta_1 t + \gamma^2 ax^2)z \\ &= (-\alpha_2 \gamma rbx + \gamma \alpha_1 rb + \gamma^2 ax^2)z \\ &= \alpha_2 \gamma \alpha_1 rt - \alpha_2 \gamma^2 rbx^2 - \gamma \alpha_1 \alpha_2 rt + \\ &\quad + \gamma^2 \alpha_1 rbx - \gamma^2 \alpha_1 rbx + \gamma^2 \alpha_2 rbx^2 \\ &= 0; \end{aligned}$$

$$\begin{aligned} axz^3 &= (\alpha_1 rb - \alpha_3 rbx^2 + \beta_1 t + \gamma ax^2)z^2 \\ &= (\alpha_1 rb + \gamma ax^2)z^2 \\ &= (-\alpha_1 \alpha_2 rt + \alpha_1 \gamma rbx - \gamma \alpha_1 rbx + \gamma \alpha_2 rbx^2)z \\ &= 0; \end{aligned}$$

$$ax^2 z^3 = (-\alpha_1 rbx + \alpha_2 rbx^2)z^2 = 0;$$

$$bz^3 = (-\alpha_2 t + \gamma bx)z^2 = 0;$$

$$bxz^3 = (-\alpha_1 t + \gamma bx^2)z^2 = 0;$$

$$\begin{aligned}
xz^3 &= (-\alpha_1 ax - \alpha_2 ax^2 - \beta_1 bx - \beta_2 bx^2)z^2 \\
&= (-\alpha_1 ax - \alpha_2 ax^2)z^2 \\
&= (-\alpha_1^2 rb + \alpha_1 \alpha_3 rbx^2 - \alpha_1 \beta_1 t + \\
&\quad -\alpha_1 \gamma ax^2 + \alpha_2 \alpha_1 rbx - \alpha_2^2 rbx^2)z \\
&= (-\alpha_1^2 rb - \alpha_1 \gamma ax^2 + \alpha_2 \alpha_1 rbx)z \\
&= \alpha_1^2 \alpha_2 rt - \alpha_1^2 \gamma rbx + \alpha_1^2 \gamma rbx + \\
&\quad -\alpha_1 \gamma \alpha_2 rbx^2 - \alpha_2 \alpha_1^2 rt + \alpha_2 \alpha_1 \gamma rbx^2 \\
&= 0.
\end{aligned}$$

□

We finally prove the nilpotency of  $L(r)$ .

**Theorem 6.8.**  $L(r)$  is nilpotent of class 3 if  $r = 0$  and of class 5 if  $r \neq 0$ .

*Proof.* Let  $r = 0$ . Then  $Z(L) = Fax^2 + Fbx^2 + Ft$  by Lemma 6.3. Moreover

$$L^2 = Lx + Ft \quad \text{and} \quad L^3 = Lx^2 + Ft = Z(L),$$

so that  $L(0)$  is nilpotent of class 3.

Assume  $r \neq 0$ . Then

$$\begin{aligned}
L^2 &= \langle b, ax, bx, ax^2, bx^2, t \rangle, & L^3 &= \langle b, bx, ax^2, bx^2, t \rangle \\
L^4 &= \langle bx, bx^2, t \rangle, & L^5 &= \langle bx^2, t \rangle, & L^6 &= \{0\}.
\end{aligned}$$

This proves that  $L(r)$  is nilpotent of class 5. □

The parameter  $r \in F$  is not unique. Recall that  $r = (a, ax^2(ax))$ . Now  $Z_3(L) = (L^4)^\perp = \langle b, bx, ax^2, bx^2, t \rangle$ . Let

$$\bar{a} = \alpha_1 a + \beta_1 ax + \gamma x + u \quad \text{and} \quad \bar{x} = \alpha_2 a + \beta_2 ax + \delta x + v$$

with  $u, v \in Z_3(L)$ . Tedious but direct calculations show that

$$(\bar{a}, \bar{a}\bar{x}^2(\bar{a}\bar{x})) = (\alpha_1 \delta - \alpha_2 \gamma)^3 r.$$

This implies that for  $r, s \neq 0$  we have that  $L(r) \cong L(s)$  if and only if  $r$  and  $s$  are in the same coset of the abelian group  $F^*/(F^*)^3$  (where  $F^* = F \setminus \{0\}$ ). Adding  $L(0)$ , we see that there are up to isomorphism exactly  $|F^*/(F^*)^3| + 1$  symplectic alternating algebras of dimension 8 that are nil-3 but not nil-2. If  $F$  is algebraically closed then this number is 2. As  $(\mathbb{R}^*)^3 = \mathbb{R}$ , this is also true when the underlying field is the field of real numbers. On the other hand,  $\mathbb{Q}^*/(\mathbb{Q}^*)^3$  is infinite so over the rational field we have an infinite number of examples. If  $F$  is finite then  $F^*$  is cyclic and thus  $|F^*/(F^*)^3|$  is 1 or 3 depending on whether 3 divides  $|F| - 1$  or not.



## References

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